Deliver the Vote!

Micromotives and Macrobehavior in Electoral Fraud

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[INCOMPLETE WORK IN PROGRESS, ALL COMMENTS WELCOME!]

Abstract

Most election fraud is not conducted centrally by incumbents but rather locally by a machinery consisting of hundreds of political operatives. How does an incumbent ensure that his local agents deliver fraud when needed and as much as is needed? We address this and related puzzles in the political organization of election fraud by studying the perverse consequences of two distinct incentive problems: the principal-agent problem between an incumbent and his local agents, and the collective action problem among the agents. Using the global game methodology, we show that these incentive problems result in a herd dynamic among the agents that tends to either oversupply or undersupply fraud, rarely delivering the amount of fraud that would be optimal from the incumbent’s point of view. This equilibrium dynamic predicts overwhelming victories for incumbents that are punctuated by his rare but resounding defeats and it explains why incumbents who enjoy genuine popularity often engage in seemingly unnecessary fraud. A statistical analysis of anomalies in precinct-level results of Russian legislative and presidential elections supports our key claims.

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1 Introduction

“You may have won the election, but I won the count!” was Anastasio Somoza’s rebuke to an opponent who accused him of rigging an election.1 A burgeoning literature depicts the many ways in which incumbents attempt to “win the count” and conducts increasingly sophisticated analyses of their detection and deterrence.2 Yet while “winning the count” may be directed and facilitated from above, its execution is primarily local. Most frequent forms of election fraud – ballot box stuffing, multiple voting, voter intimidation, or the falsification of counts – are ultimately executed at the level of individual polling stations, not by the incumbent himself, but rather a machinery that typically consists of hundreds of political operatives, party members, and state employees.

In spite of the rich descriptive accounts of such local-level fraud in qualitative and historical literature, most formal and analytical research on election fraud treats its political organization and execution as unproblematic. The incumbent’s machinery of manipulation is assumed to act as a unified political actor, under the incumbent’s perfect political control. This approach leaves us with a number of puzzles: How does the incumbent ensure that his local agents deliver fraud when needed and exactly as much as is needed? What motivates local agents to engage in fraud when doing so may result in criminal prosecution and conviction? Why does locally-conducted electoral fraud succeed in delivering a victory in some elections but fail in others?

In this paper, we demonstrate that incentive problems in the political organization and execution of election fraud have far-reaching implications for its conduct, success, and

2[See the discussion of our contribution below for specific references.] On the conduct and forms of election fraud, see Birch (2011, Chapter 2), Lehoucq and Molina (2002), Lehoucq (2003), Magaloni (2010), Schedler (2012), and Simpser (2013). On its detection and deterrents, see Alvarez, Hall, and Hyde (2008), Hyde (2011a), Hyde and Marinov (2009), Ichino and Schüdeln (2012), Mebane and Kalinin (2009), Myagkov, Ordeshook, and Shakin (2009), and Sjoberg (2013).
empirical fingerprints. Underlying the above puzzles are two related but distinct incentive
problems: the principal-agent problem between an incumbent and his local agents, and the
collective action problem among the agents. At the heart of the principal-agent problem is
a conflict of interest between the incumbent and his agents about when to engage in fraud
and how much of it to conduct. The incumbent’s preferences were eloquently summarized
in John F. Kennedy’s facetious response to questions about the role of his farther’s wealth
in his political success: “I have just received the following wire from my generous daddy. It
says, ‘Dear Jack, don’t buy a single vote more than is necessary. I’ll be damned if I’m going
to pay for a landslide!’”³ That is, even those incumbents who are willing to engage in
fraud if it is needed for a victory want to avoid unnecessary, excessive fraud that will only
raises suspicions. Most agents, meanwhile, prefer to conduct fraud when it carries the least
risk – when the incumbent’s victory is assured and the agents’ actions are unlikely to be
investigated. Agents are most reluctant to engage in fraud when the incumbent’s victory is
in doubt and they worry about being prosecuted if the challenger wins the election. Thus
agents are least willing to engage in fraud precisely when the incumbent needs it most!

This principal-agent problem between the incumbent and his agents is compounded by
a collective action problem among the agents. It is most pronounced when the incumbent
narrowly trails the challenger. In these scenarios, the incumbent’s agents understand that,
if only enough of them engaged in fraud, they could secure the incumbent’s victory. At the
same time, however, each agent’s doubts about other agents’ actions lead her to question
the prudence of her own engagement in fraud. Hence even when fraud could assure the
incumbent’s victory, the agents’ fear of its ultimate failure may turn it into a self-fulfilling
prophecy.

In order to rigorously examine the interplay between principal-agent and collective

action problems in the political organization and execution of election fraud, we develop a formal model with two key, novel features. First, the incumbent does not engage in fraud directly but instead depends on the illicit collaboration of a large number of local agents who must be motivated by the promise of a reward. This is a departure from existing formal research where the incumbent’s machinery of fraud is assumed to act as a unitary actor (Egorov and Sonin 2012; Fearon 2011; Gandhi and Przeworski 2011; Little 2012; Rozenas 2013; Simpser 2013; Svolik and Chernykh 2013). A key aspect of this departure concerns the contingency of the agents’ rewards – as well as their punishment – on the incumbent’s political survival: Each agent understands that she will obtain the promised reward only if the incumbent is reelected and may face prosecution if she engaged in fraud but the incumbent is ultimately defeated.4

The second key feature of our formalization concerns the limited information available to the incumbent and his local agents when they are deciding on whether to engage in fraud. The difficulties that incumbents in hybrid regimes face when gauging their genuine popularity have been highlighted in classic accounts of incentives for “preference falsification” under authoritarianism (Kuran 1991; Wintrobe 1998), in research on political attitude formation in new and transitioning democracies (Kitschelt et al. 1999; Grzymała-Busse 2002; Tucker 2006), and in the more recent formal research on electoral manipulation and democratization (Little 2013; Rozenas 2013; Svolik 2013). The novel feature of our model is in how the structure of this information paucity is tailored to the context of electoral manipulation: While both the incumbent and his local agents have imperfect information about the incumbent’s genuine popularity, each local agent’s

4Such politically contingent rewards play a key role in Gehlbach and Simpser’s (2011) analysis of bureaucratic incentives under authoritarianism. In their setting, an incumbent manipulates the public perception of his popularity in order to motivate a single bureaucratic agent to exert costly effort that contributes to his survival. The focus of our analysis is how the anticipation of the likely outcome of an election motivates (or fails to motivate) a large number of the incumbent’s agents to conduct risky fraud on his behalf, resolve their collective action problem, and thus to secure his victory (or to seal his defeat.)
information is much more precise than the incumbent’s but at the same time confined to her own precinct.

The chief macro-political consequence of these two novel features is a herd dynamic among the agents that tends to result in either overwhelming victories for the incumbent or, less often, his resounding defeats. We obtain this prediction by a natural application of the global game approach to the analysis of collective action problems (Carlsson and van Damme 1993; Morris and Shin 2003). Its key advantage is to transform a setting that would otherwise suffer from a multiplicity of equilibria with contradictory predictions into one with a unique, tractable, and politically intuitive equilibrium. In our setting, the agents’ incentives result in a unique tipping-point equilibrium according to which an agent engages in fraud only if her local, private perception of the incumbent’s popularity is above a certain threshold. The intuition is as follows: The higher the incumbent’s genuine popularity in an agent’s precinct, the more popular she infers the incumbent is nationwide; consequently, she anticipates that fewer agents need to engage in fraud in order to secure the incumbent’s victory, which in turn lowers her own risk of engaging in fraud. Thus while never observed perfectly by either the incumbent or the agents, the incumbent’s genuine nationwide popularity ends up playing a central role in coordinating the agents’ attempts to resolve their collective action problem.

The perverse consequence of such individual-level incentives is a herd dynamic at the aggregate level: Jointly, agents will tend to either oversupply or undersupply fraud, rarely delivering the amount of fraud that would be optimal from the incumbent’s point of view. At one extreme, when the incumbent is unpopular and needs fraud most, agents will tend to underdeliver it; at the other extreme, agent will deliver excess fraud when not needed at

\footnote{See Boix and Svolik (2013), Bueno de Mesquita (2010), Edmond (2013), Little (2012), and Shadmehr and Bernhardt (2011) for applications of global games and related techniques to the analysis of collective action problems in protests, repression, regime change, and authoritarian power-sharing.}
all. From the incumbent’s point of view, the aggregate amount of fraud will approximate its optimal level only in genuinely close elections. It is only in such elections that fraud will be both politically decisive for the incumbent’s victory and succeed in securing it.

Our analysis of microincentives in the political organization of election fraud improves our understanding of the resulting macrobehavior in a number of ways. First, the equilibrium dynamic that we just described helps us understand the puzzling, often contradictory accounts of incumbents who enjoy genuine popularity and at the same time engage in seemingly unnecessary fraud. In a seminal analysis of the Institutional Revolutionary Party’s (PRI) demise in Mexico, Magaloni (2006) for instance observes that, while certainly present, fraud only served to embellish the already impressive popularity that the PRI enjoyed before the 1980s (see also Greene 2007; Simpser 2013). Similarly, students of contemporary Russia are puzzled by the embarrassingly obvious fingerprints of fraud in elections that the United Russia party, and especially Putin and Medvedev, could have won cleanly. According to a leading explanation for these perplexing outcomes, inflated margins of victory serve to signal the incumbent’s invincibility and thus deter potential defectors and challengers (Magaloni 2006; Simpser 2013). Our analysis suggest an alternative mechanism. Rather than an intentional strategy, overwhelming incumbent victories may be the unintended byproduct of the principal-agent and collective action problems in the political organization of election fraud: Because individual local agents are most willing to conduct fraud when it carries the least risk – when the incumbent is genuinely popular – we should not be surprised to observe genuine popularity go hand in hand with excessive fraud at the aggregate level.

This logic also helps us understand why local-level fraud, even when encouraged by the

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incumbent, sometimes fails to secure his re-election. Because fraud is by definition illegal, the incumbent’s capacity to motivate agents to engage in fraud on his behalf is limited to the tacit promise of a reward upon his reelection. Such politically contingent inducements, however, are least effective precisely when the incumbent needs the agents’ collaboration most – when he lacks genuine popularity. Our analysis of the ensuing collective action problem highlights how individual agents’ worries about the incumbent’s eventual defeat reverberate among them and, if sufficiently pronounced, multiply into an avalanche of defections from the incumbent. Thus even though any single agent’s decision to refrain from fraud may be inconsequential at the national level, a minor decline in the incumbent’s popularity may instigate a large aggregate shift from close to universal participation in fraud to an almost complete abstention. Agents’ fears of the incumbent’s defeat become a self-fulfilling prophecy.

This reasoning suggests a mechanism of fraud deterrence that has not been explored by existing theoretical research but is implicit in most recent empirical research. Extant analytical treatments of election fraud focus on the threat of a post-election protest or violence as the chief deterrent against fraud (Fearon 2011; Little 2012; Przeworski 2011; Svolik and Chernykh 2013; Tucker 2007). By contrast, our arguments highlight that a major reason for the failure of local-level fraud may be the incumbent’s inability to muster the machinery of fraud in the face of his declining popularity. This focus on incentives faced by local-level agents parallels empirical research on election monitoring, the deterrent effect of which is also hypothesized to occur at the level of individual polling stations (Hyde 2008; Ichino and Schüdeln 2012). Our results, however, suggest that the direct effect of local-level deterrents – whereby they raise the risk of engaging in fraud for individual

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agents who are being monitored – may not be the most consequential one.\textsuperscript{8} Rather, the primary consequence of monitoring may be to heighten the collective action dilemma among all agents, even those who are not being monitored: When monitoring occurs, all agents anticipate that much greater efforts must be exerted at non-monitored polling stations in order to secure the incumbent’s victory. To our knowledge, such indirect, systemic consequences of local fraud deterrents have not yet been examined either empirically or theoretically.

An improved understanding of the microincentives faced by the agents who ultimately execute fraud also helps us anticipate its empirical fingerprints. The prevailing approach to fraud detection focuses on the identification of statistical anomalies in voting or turnout outcomes.\textsuperscript{9} Our model clarifies that anomalies indicative of fraud may be the unintended consequence of incentive problems in the political organization of local-level fraud and predicts a specific pattern that such anomalies should follow: Their occurrence across precincts should not be uniform but rather increasing in both the incumbent’s genuine popularity and his vote share. Yet at the same time, such anomalies alone do not imply that the incumbent stole an election that would have otherwise been won by the challenger. In fact, the fingerprints of fraud may be most pronounced precisely when fraud is not politically decisive.

In the next section, we present our model of the political organization of electoral fraud. Proofs of all technical results as well as the discussion of alternative parameterizations of our information structure can be found in the supplementary appendix.

\textsuperscript{8}After all, only a small fraction of polling stations is visited by election observers during any single election; see e.g. Hyde (2011b).

Consider the following electoral manipulation game between an incumbent and a continuum of his agents. Each agent $i$ operates in one among a continuum of precincts of equal size and decides whether to engage in fraud on behalf of the incumbent at the time of the election. We denote agent $i$’s engagement or not in fraud by $a_i = \{f, n\}$, respectively. The incumbent, however, does not observe whether any agent engaged in fraud; he only observes the *precinct-level election result* $R_i$ – his share of the vote in agent $i$’s precinct.

Before the election, therefore, the incumbent promises each agent a reward (a higher salary, promotion, perks) commensurate with the election result in her precinct. More precisely, each agent obtains the payoff $wR_i$ after the incumbent’s victory, where $w > 0$ and we refer to is as the *reward factor*.

The precinct-level election result $R_i$ depends on the incumbent’s genuine *precinct-level popularity* $S_i$ and whether the agent engaged in fraud on behalf of the incumbent, $a_i = \{f, n\}$, as follows:

$$R_i = \begin{cases} S_i + F & \text{if } a_i = f; \\ S_i & \text{if } a_i = n. \end{cases}$$

Above, the parameter $F$, $0 < F < \overline{F}$, denotes the share of the precinct-level election result due to the agent’s fraud and we interpret it as a measure of the precinct agents’ *fraud capacity*.\(^{10}\)

Crucially, the agents obtain the reward $wR_i$ only if the incumbent is re-elected. By contrast, if the incumbent loses, each agent’s payoff depends on whether she engaged in

\(^{10}\)This “additive” assumption about fraud production is only one – and an intentionally simple one – among several plausible ways of formalizing fraud production. Letting $R_i = (1 + F)S_i$ or $R_i = \frac{(1 + F)S_i}{(1 + F)S_i + (1 - S_i)}$ results in qualitatively identical insights but less transparent algebra. In the supplementary appendix, we derive $\overline{F}$, the maximum admissible value of $F$ that is implied by our informational assumptions and the global game framework (see below); $\overline{F} = \frac{1}{2}(1 - 4\epsilon)$ for $\epsilon < \frac{1}{4}$. 

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Figure 1: Agent \( i \)'s payoffs as a function of her fraud decision \( a_i \) and the election result \( R \)

FRAUD at the time of the election. If she did not engage in fraud, she obtains the payoff 0. If she did engage in fraud, the agent obtains the payoff \(-cF\), where \( c > 0 \) stands for the political cost of fraud. It reflects the potential investigation of allegations of fraud in the agent’s precinct after the challenger takes office, possibly resulting in the agent’s criminal prosecution and conviction. Thus we are effectively assuming that the likelihood of a conviction or the severity of punishment are increasing in \( F \), the share of the precinct-level election result due to the agent’s fraud. \(^{11}\)

These payoffs are summarized in Figure 1. We see that each agent’s incentive to engage in fraud depends on her expectation about the national-level election result \( R \). If the agent expects the incumbent to win, \( R \geq \frac{1}{2} \), then she prefers to conduct fraud since \( w(S_i + F) > wS_i \). If, on the other hand, the agent expects the incumbent to lose, \( R < \frac{1}{2} \), then she prefers to play fair since \(-cF < 0\). \(^{12}\)

The overall election result \( R \) depends on both the incumbent’s genuine popularity and the actions of the agents. More specifically, the incumbent’s popularity at the time of the election \( \theta \), \( 0 < \theta < 1 \), corresponds to the fraction of the electorate that actually voted for the incumbent. We assume that \( \theta \) is not perfectly known by any of the players but is commonly believed to be uniformly distributed on the unit interval \((0, 1)\). Instead, each agent privately observes the fraction \( S_i \) of her precinct that voted for the incumbent, which is correlated with \( \theta \) in the following way: \( S_i \) is uniformly distributed on the interval \( [0, 1] \).

\(^{11}\)We may also interpret \( c \) less politically, as the agent’s costly effort. In that case, \(-cF\) should also be included in the top-left cell of the payoff table in Figure 1.

\(^{12}\)Thus we implicitly assume a two-candidate competition. We break ties in favor of the incumbent because it simplifies notation later; this is politically inconsequential.
$(\theta - \epsilon, \theta + \epsilon), 0 < \epsilon < \tau$.\textsuperscript{13} We think of $\epsilon$ as “small” and interpret it as a measure of heterogeneity in the incumbent’s support across precincts.\textsuperscript{14} Thus when each agent decides whether to engage in fraud on behalf of the incumbent, she has only imperfect information about the incumbent’s genuine, national-level popularity.\textsuperscript{15}

Because we model the incumbent’s agents as atomless players on a continuum of precincts, any single agent’s decision to engage in fraud on behalf of the incumbent will be inconsequential at the national level. Jointly, however, the agents’ actions affect the election result as the overall election result $R$ amounts to

$$E[R] = \int_{\theta-\epsilon}^{\theta+\epsilon} \frac{1}{2\epsilon} (S_i + 1_{\{a_i=f\}}F) \, dS_i = \theta + \phi F. \quad (1)$$

Above, $E[R]$ denotes the expected value of $R$, $1_{\{a_i=f\}}$ is an indicator function that equals 1 if agent $i$ engaged in fraud and 0 otherwise, and $\phi$ is the fraction of agents that engaged in fraud. We think of our assumption of atomless agents along a continuum of equally-sized precincts as capturing a country with a large number of precincts that are small relative to the country as a whole.\textsuperscript{16} This approximation works well since the national-level election result in such a country is effectively the mean of precinct-level results, $R = \frac{1}{N} \sum_{i=1}^{N} R_i$. By

\textsuperscript{13}The possibility of $S_i < 0$ and $S_i > 1$ when $\theta$ is within an $\epsilon$ distance of the boundaries 0 and 1, respectively, is irrelevant for the strategic analysis that follows (and could be avoided by letting $S_i$ be uniformly distributed on the intervals $(0, \theta + \epsilon)$ and $(\theta - \epsilon, 1)$ when $\theta$ is within an $\epsilon$ distance of 0 and 1, respectively.) The advantage of an information structure based on the Uniform distribution is the availability of closed form solutions for key quantities in our analysis. In section 2.2 and the supplementary appendix, we present results based an alternative information structure according which $\theta$ and $S_i$ follow the (appropriately transformed) Normal distribution.

\textsuperscript{14}In the supplementary appendix, we derive $\tau$, the maximum admissible value of $\epsilon$ that is implied by our informational assumptions, $\tau = \frac{1}{4}(1 - 2F)$ for $F < \frac{1}{2}$.

\textsuperscript{15}Note that while $S_i$ is informative about $\theta$, the agents lack a common knowledge of $\theta$ for an arbitrarily small $\epsilon$. For a discussion of this feature of global games, see Morris and Shin (2003).

\textsuperscript{16}Our results extend to a setting with a finite number of agents; c.f. Morris and Shin (2003, Appendix B).
the law of large numbers, this mean converges in probability to (1),

\[ R = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (S_i + 1\{a_i = f\}F) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} S_i + F \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} 1\{a_i = f\} = \theta + \phi F. \]

The national-level election outcome \( R \) thus depends on the precinct agents’ fraud capacity \( F \), the fraction of agents engaging in fraud \( \phi \), and the incumbent’s election-day popularity \( \theta \). If \( \theta \geq \frac{1}{2} \), then \( R \geq \frac{1}{2} \) and the incumbent wins the election regardless of the agents’ actions. If \( \theta < \frac{1}{2} - F \), on the other hand, then the incumbent will be defeated even if all agents conducted fraud on his behalf, \( R < \frac{1}{2} \). Only when \( \frac{1}{2} - F \leq \theta < \frac{1}{2} \) does the election outcome depend on the fraction of agents \( \phi \) engaging in fraud. In turn, if the agents were able to observe \( \theta \) perfectly, they would all refrain from fraud when \( \theta < \frac{1}{2} - F \), engage in fraud if \( \theta \geq \frac{1}{2} \), and condition their actions on the actions of others when \( \frac{1}{2} - F \leq \theta < \frac{1}{2} \). In the latter, politically most interesting case, all agents engaging in fraud and refraining from fraud both constitute a Nash equilibrium.

This indeterminacy disappears in our setting where agents do not directly observe the incumbent’s national-level popularity \( \theta \). While each agent’s precinct-level result \( S_i \) is only an imperfect signal of \( \theta \), some values of \( S_i \) allow him to perfectly infer the outcome of the election. More specifically, our model for the distribution of \( S_i \) implies that if \( S_i < \frac{1}{2} - F - \epsilon \), then \( \theta < \frac{1}{2} - F \) and thus \( R < \frac{1}{2} \). On the other hand, if \( S_i \geq \frac{1}{2} + \epsilon \), then \( \theta \geq \frac{1}{2} \) and thus \( R \geq \frac{1}{2} \). For these values of \( S_i \), therefore, each agent optimally refrains from and engages in fraud, respectively. But when \( \frac{1}{2} - F - \epsilon \leq S_i < \frac{1}{2} + \epsilon \), agent \( i \)'s optimal action depends on her inference about other agents’ precinct-level results and actions.

Consider therefore Bayesian Nash equilibria in threshold strategies \( \sigma(S_i) \) for agents with precinct-level results on the interval \( \frac{1}{2} - F - \epsilon \leq S_i < \frac{1}{2} + \epsilon \). According to these strategies, agent \( i \) engages in fraud if and only if the incumbent’s popularity in her precinct
is at least some threshold value $S^*$,

$$
\sigma(S_i) = \begin{cases} 
\text{engage fraud, } a_i = f, & \text{if } S_i \geq S^*; \\
\text{do nothing, } a_i = n, & \text{if } S_i < S^*. 
\end{cases}
$$

We will refer to $S^*$ as the agents’ fraud threshold.

When an agent whose precinct-level result is $S_i$ engages in fraud, she expects the payoff

$$
\Pr \left[ R \geq \frac{1}{2} \mid S_i \right] w(S_i + F) - \Pr \left[ R < \frac{1}{2} \mid S_i \right] cF. \tag{2}
$$

Meanwhile, the agent’s expected payoff from doing nothing is

$$
\Pr \left[ R \geq \frac{1}{2} \mid S_i \right] wS_i. \tag{3}
$$

According to the threshold strategy $\sigma(S_i)$, the threshold agent in whose precinct the incumbent’s popularity is $S_i = S^*$ must be indifferent between engaging in fraud and doing nothing. Letting $S_i = S^*$ and equating (2) to (3), we see that the following indifference condition holds for the threshold agent:

$$
\Pr \left[ R < \frac{1}{2} \mid S_i = S^* \right] = \frac{w}{c + w}. \tag{4}
$$

The indifference condition in (4) highlights the central role that each agent’s expectation about the outcome of the election plays in her decision to engage in fraud. The smaller the reward factor $w$, the stronger must be the threshold agent’s expectation that the incumbent will win. This will occur when the fraction of agents that engage in fraud $\phi$
satisfies the following *majority condition*

\[ R \geq \frac{1}{2} \text{ or equivalently } \theta + \phi F \geq \frac{1}{2}. \]

We may therefore refer to the value of \( \phi \) at which the incumbent wins by a bare majority as the *majority threshold* \( \phi^* \),

\[ \phi^* = \frac{1}{2} - \theta. \]

According our assumptions about the distribution of \( S_i \) and the threshold strategy \( \sigma(S_i) \), the fraction of agents that engage in fraud in equilibrium is

\[ \phi = \frac{(\theta + \epsilon) - S^*}{2\epsilon}. \]

In turn, the majority threshold implies that the threshold agent’s belief that the incumbent will lose the election is

\[
\Pr \left[ R < \frac{1}{2} \mid S_i = S^* \right] = \Pr \left[ \phi < \phi^* \mid S_i = S^* \right]
= \Pr \left[ \frac{(\theta + \epsilon) - S^*}{2\epsilon} < \phi^* \right]
= \Pr \left[ \theta < S^* + 2\epsilon \phi^* - \epsilon \right].
\] (5)

Substituting the majority threshold \( \phi^* \) into (5), we obtain

\[
\Pr \left[ R < \frac{1}{2} \mid S_i = S^* \right] = \Pr \left[ \theta < \frac{FS^* - \epsilon S^* + \epsilon}{F + 2\epsilon} \right].
\]

Given that \( \theta \) and \( S_i \) are uniformly distributed on the intervals \((0, 1)\) and \((\theta - \epsilon, \theta + \epsilon)\), respectively, the threshold agent believes that the incumbent’s popularity \( \theta \) is uniformly
distributed on the interval \((S^* - \epsilon, S^* + \epsilon)\). In turn,

\[
\Pr \left[ R < \frac{1}{2} \mid S_i = S^* \right] = \frac{FS^* - \epsilon S^* + \epsilon}{F + 2\epsilon} - \frac{(S^* - \epsilon)}{2\epsilon} = \frac{1}{2} - \frac{S^* + \epsilon}{F + 2\epsilon}.
\]

Substituting (6) into the indifference condition in (4), we see that the agents’ fraud threshold must be

\[
S^* = \frac{1}{2} - F \left( \frac{w}{c + w} + \frac{c - w}{2\epsilon} \right).
\]

Jointly, the fraud threshold \(S^*\) and majority threshold \(\phi^*\) imply the existence of a popularity threshold \(\theta^*\) such that, in equilibrium, the incumbent loses the election if \(\theta < \theta^*\) and wins the election if \(\theta \geq \theta^*\). That is, when the incumbent’s genuine popularity is exactly at the popularity threshold \(\theta^*\), he wins by a bare majority,

\[
\theta^* = \frac{1}{2} - F \left( \frac{w}{c + w} + \frac{c - w}{2\epsilon} \right) \quad \text{and} \quad \phi^* = \frac{(\theta^* + \epsilon) - S^*}{2\epsilon}.
\]

Substituting \(S^*\) above and solving for \(\theta^*\) and \(\phi^*\), we see that

\[
\theta^* = \frac{1}{2} - F \left( \frac{w}{c + w} \right) \quad \text{and} \quad \phi^* = \frac{w}{c + w}.
\]

These results are summarized in the following proposition.

**Proposition 1 (Collective Action and Election Fraud).** *In the unique Bayesian Nash equilibrium,*

i. agent \(i\) engages in fraud \((a_i = f)\) if \(S_i \geq S^*\) and does nothing \((a_i = n)\) otherwise,

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17This holds for the posterior density of \(\theta \mid S_i\) for the range of \(S_i\) under consideration, \(\frac{1}{2} - F - \epsilon \leq S_i < \frac{1}{2} + \epsilon\). The posterior density of \(\theta \mid S_i\) within a \(2\epsilon\) distance of the boundaries 0 and 1 is different; see the supplementary appendix for details.
ii. the incumbent wins the election \( R \geq \frac{1}{2} \) if \( \theta \geq \theta^* \) and is defeated \( R < \frac{1}{2} \) otherwise,

iii. the fraction of agents that engage in fraud when the incumbent barely wins the election \( R = \frac{1}{2} \) is \( \phi^* \),

where the fraud, popularity, and majority thresholds are, respectively

\[
S^* = \theta^* + \frac{c - w}{c + w}, \quad \theta^* = \frac{1}{2} - F\phi^*, \quad \text{and} \quad \phi^* = \frac{w}{c + w}.
\]

**Proof.** Follows from the text. See the supplementary appendix for the derivation of the upper bounds on \( F \) and \( \epsilon \).

\[\Box\]

### 2.1 Comparative Statics and Political Implications

In order to highlight the political implications of our results, consider an illustration based on the parameters \( w = 3, \ c = 1, \ F = \frac{2}{10}, \ \epsilon = \frac{1}{10}, \) which yield \( S^* = 0.3, \ \theta^* = 0.35, \) and \( \phi^* = 0.75. \) That is, in equilibrium, agents engage in fraud only if the incumbent’s popularity in their precinct is greater than 30% and fraud secures the incumbent’s victory if his national-level popularity is greater than 35%, or equivalently, when at least three-fourths of agents participate in fraud. Figure 2 employs these values to plot the equilibrium election result \( R^* \) as a function of the incumbent’s genuine popularity \( \theta. \)

We see that the agents’ equilibrium behavior can be partitioned into four qualitatively distinct intervals over \( \theta. \) At very low levels of the incumbent’s genuine popularity, \( 0 < \theta < S^* - \epsilon, \) no agent observes a precinct-level popularity high enough to warrant engaging in fraud and all agents correctly anticipate the incumbent’s defeat. When \( S^* - \epsilon \leq \theta < \theta^*, \) strategic miscoordination occurs: if enough agents engaged in fraud, they could ensure the incumbent’s victory for some values of \( \theta \) on this interval (since \( \frac{1}{2} - F = 0.3 \)), but because the incumbent’s popularity is too low, an insufficient number of
agents ends up engaging in fraud. By contrast, when $\theta^* \leq \theta < \frac{1}{2}$, successful coordination occurs: enough agents engage in fraud to secure an *undeserved victory* for the incumbent. If we take the share of the election result that is due to fraud as a measure of such undeservedness, then the incumbent’s victory is most undeserved when $\theta = S^* + \epsilon$ and all agents engage in fraud. Finally, when $\frac{1}{2} < \theta < 1$, fraud occurs but it is *unnecessary*: the incumbent is popular enough to win without fraud. In fact, when $\theta \geq \frac{1}{2} + \epsilon$ all agents are aware that the incumbent will prevail without their complicity; they nonetheless engage in fraud because it boosts the election result in their precincts and thus leads to a higher reward. A perverse consequence of these incentives are national-level election results exceeding 100% at high values of $\theta$.

This conflict between the incumbent’s needs and the agents’ equilibrium behavior is
Figure 3: The association between the incumbent’s genuine, national-level popularity $\theta$ (bottom axis), the equilibrium election result $R^*$ (top axis), the equilibrium level of fraud $F^*$ (solid black line), and the level of fraud needed for a victory $\hat{F}$ (dashed gray line) illustrated in Figure 3. The bottom axis denotes the incumbent’s national-level popularity $\theta$; the top axis denotes the equilibrium election result $R^*$; the dashed gray line plots the level of fraud $\hat{F}$ that the incumbent needs for a victory; and the solid black line plots the level of fraud $F^*$ that occurs in equilibrium. When $\theta < \theta^*$, the incumbent needs more fraud for a victory than the agents deliver, $F^* < \hat{F}$. When $\theta \geq \theta^*$, on the other hand, the agents engage in a level of fraud that is unnecessary from the incumbent’s point of view, $F^* > \hat{F}$, jointly delivering up to 20% in excess of what the incumbent needs. Only exactly at $\theta^*$ are the incumbent’s needs and the agents’ equilibrium behavior in balance: This is when the incumbent needs the national level of fraud to add up to 15% ($\phi^* F = 0.15$) and just the right fraction of agents – three-fourths ($\phi^* = 0.75$) – delivers it.
Precinct-level results thus play a central role in forming the agents’ beliefs about the incumbent’s national-level popularity and, in turn, the degree of agents’ coordination needed for the incumbent’s victory. But the precise value of the thresholds \( S^*, \theta^*, \) and \( \phi^* \) is also shaped by the parameters \( F, w, c, \epsilon \). An increase in the fraud capacity \( F \) and the reward factor \( w \) lowers the fraud and popularity thresholds \( S^* \) and \( \theta^* \) and thus the region along which the incumbent secures an undeserved victory; the cost of engaging in fraud \( c \) has the opposite effect. That is, the greater the amount of fraud that each agent can produce within her precinct, the lower the demands on the agents’ coordination.

Meanwhile, the greater the agents’ compensation for precinct-level results, the greater the risk that each agent is willing to take when engaging in fraud. The opposite holds for the cost parameter \( c \). At the lowest values of \( F \) and \( w \) (\( F = 0 \) or \( w = 0 \)), no fraud occurs in equilibrium when it is actually needed by the incumbent as \( S^* = \frac{1}{2} + \epsilon, \theta^* = \frac{1}{2}, \) and \( \phi^* = 0. \) Intuitively, no agent is willing to risk prosecution when there is no way to inflate the incumbent’s vote share or when there is no personal benefit from doing so.

Crucially, while a greater reward factor \( w \) lowers the fraud and popularity thresholds \( S^* \) and \( \theta^* \), it cannot eliminate the collective action problem among the agents.\(^\text{18}\) Observe that as \( w \) tends to infinity, \( \lim_{w \to \infty} \phi^* = 1 \), and in turn, \( \lim_{w \to \infty} S^* = \frac{1}{2} - F - \epsilon \) and \( \lim_{w \to \infty} \theta^* = \frac{1}{2} - F \), but

\[
S^* > \frac{1}{2} - F - \epsilon \quad \text{and} \quad \theta^* > \frac{1}{2} - F \quad \text{for any} \quad w > 0.
\]

That is, even when the agents’ compensation is arbitrarily large, there will be values of the incumbent’s popularity at which the agents could deliver the incumbent’s victory by

\(^{18}\)The converse holds for the cost parameter \( c \): while a greater cost of engaging in fraud raises the fraud and popularity thresholds \( S^* \) and \( \theta^* \), it cannot entirely prevent fraud from succeeding in equilibrium; \( S^* < \frac{1}{2} + \epsilon \) and \( \theta^* < \frac{1}{2} \) for any \( c > 0. \)
conducting fraud but will fail to do so out of the fear that an insufficient fraction of them will engage in fraud.

Finally, the heterogeneity of the incumbent’s support across precincts $\epsilon$ determines (jointly with $F$) the size of the interval along which agents face a strategic coordination problem.\footnote{The effect of $\epsilon$ on $S^*$ is less straightforward; see the supplementary appendix.} When the incumbent loses the election within this interval, his defeat may be accompanied by a disproportionately large shift in the election result. Specifically, a $2\epsilon$ decline in the incumbent’s popularity on the interval $(S^* - \epsilon, S^* + \epsilon)$ is associated with a $2\epsilon + F$ decline in the election result, from $S^* + \epsilon + F$ to $S^* - \epsilon$. At our parameter values, a decline in the incumbent’s genuine popularity from 40\% to 20\% corresponds to the difference between an electoral victory with a vote share of 60\% and a defeat with a vote share of 20\%. Hence a moderate shift in the incumbent’s genuine popularity may result in an overwhelming defeat of the incumbent.

### 2.2 An Alternative Information Structure: The Normal Model

The key advantage of the Uniform information structure that we have employed so far is the availability of closed form solutions for the fraud, popularity, and majority thresholds, and in turn, the ease of conducting comparative statics. While simplifying our analysis, the Uniform model makes two strong assumptions. First, the common uniform prior for $\theta$ on the interval $(0, 1)$ effectively assumes complete ignorance among the agents about the incumbent’s national-level popularity prior to observing their precinct-level signals $S_i$. Second, the distribution of $S_i$ on the interval $(\theta - \epsilon, \theta + \epsilon)$ implies an unrealistically sharp boundary between those regions of $\theta$ where the signals $S_i$ occur with a positive probability and those where they do not occur at all.

In order to assess the robustness of our results, we examine an alternative information structure.
structure which relaxes both of the above assumptions. Specifically, we let the incumbent’s popularity \( \theta \) and the agents’ signals \( S_i \) follow the probit-transformed Normal distribution. This keeps the support of \( \theta \) and \( S_i \) on \((0, 1)\) while allowing for prior beliefs about \( \theta \) of an arbitrary mean and precision and keeping a positive amount of uncertainty along the support \((0, 1)\) of \( \theta \) and \( S_i \). This formulation not only confirms our key insights but it also highlights the sharp consequences that shifts in the agent’s perceptions of the incumbent’s popularity may have when the precision of the signal \( S_i \) is high.

Suppose, therefore, that instead of being distributed uniformly, we let \( \theta' \), the probit-transformed version of \( \theta \), follow the Normal distribution with mean \( \theta'_0 \) and variance \( \sigma^2_0, \mathcal{N}(\theta'_0, \sigma^2_0) \), and we assume that \( S'_i \) (the probit-transformed version of \( S_i \)) follows the Normal density with the mean \( \theta' \) and variance \( \sigma^2, \mathcal{N}(\theta', \sigma^2) \). Paralleling our earlier interpretation of \( \epsilon \), we think of the variance \( \sigma^2 \) of \( S'_i \) as a metric of heterogeneity in the incumbent’s support across precincts. We translate \( \theta' \) and \( S'_i \), whose support is on \((-\infty, \infty)\) onto \((0, 1)\), the natural interval \( \theta \) and \( S_i \), via the probit link, \( \theta' = \Phi^{-1}(\theta) \) and \( S'_i = \Phi^{-1}(S_i) \).

This information structure implies that the incumbent is genuinely supported by a majority of the electorate, \( \theta \geq \frac{1}{2} \), when \( \theta' \geq 0 \). The fraction of agents \( \phi \) that engage in fraud in equilibrium corresponds to one minus the cumulative distribution function of the \( \mathcal{N}(\theta'', \sigma^2) \) density evaluated at \( S'' \) and, after observing the incumbent’s (probit-transformed) popularity in her precinct \( S'_i \), agent \( i \) believes that the incumbent’s national-level (probit-transformed) popularity \( \theta' \) follows the Normal density with the mean \( \frac{\sigma^2 S'_i + \sigma^2 \theta'_0}{\sigma^2_0 + \sigma^2} \) and the variance \( \frac{\sigma^2_0 \sigma^2}{\sigma^2_0 + \sigma^2} \).20

Unlike in the case of the uniform parameterization, the threshold agent’s indifference condition for the Normal model – which we state explicitly in the supplementary appendix

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20This is the standard Bayesian inference for the Normal distribution according to which the posterior mean of \( \theta'|S'_i \) is a weighted average of the prior mean \( \theta'_0 \) and the precinct-level signal \( S'_i \) (with the weights in proportion of the prior variance \( \sigma^2_0 \) to the signal variance \( \sigma^2 \)); see e.g. Bernardo and Smith (1994, 439). We plot illustrations of the posterior density \( \theta|S_i \) in the supplementary appendix.
Figure 4: The effect of the incumbent’s genuine, national-level popularity $\theta$ on the equilibrium election result (left) and the equilibrium v. needed level of fraud (right) does not have a closed form solution. But jointly with the majority condition it results in a unique set of equilibrium fraud, popularity, and majority thresholds, which can be obtained numerically.\textsuperscript{21} Letting $\theta_0' = 0$ (which implies $\theta_0 = \frac{1}{2}$), $\sigma_0^2 = 1$, $\sigma^2 = \frac{1}{4}$ and, as previously, $w = 3$ and $F = \frac{2}{10}$, we obtain $S^* = 0.18$, $\theta^* = 0.33$, and $\phi^* = 0.67$. That is, agents engage in fraud only if the incumbent’s popularity in their precinct is greater than 18% and fraud secures the incumbent’s victory if his national-level popularity is greater than 33%, or equivalently, when at least 67% of agents participate in fraud.

Figure 4 plots the analogues of Figures 2 and 3 from our earlier discussion. For $\sigma^2 = \frac{1}{4}$, we plot the equilibrium election result and fraud as a solid black line and denote the equilibrium popularity threshold by $\theta^*_1$. We see that key conclusions from our earlier analysis remain unchanged: the equilibrium level of fraud is increasing in the incumbent’s popularity, with undeserved victories when $\theta^* \leq \theta < \frac{1}{2}$ and an over- and under-supply of fraud.

\textsuperscript{21}Uniqueness in the Normal model obtains as long as the signal $S_i$ is sufficiently precise relative to the prior belief about $\theta$; see the supplementary appendix for a precise statement.
fraud due to the equilibrium interplay of the collective action and principle-agent problems.\footnote{The comparative statics for $F$ and $w$ are also identical; see the supplementary appendix for a formal proof.} The key difference between the Uniform and the Normal models is that positive levels of fraud occur at all values of $\theta$. That is, even when $\theta$ is close to 0, there will be (a small measure of) agents who will observe precinct-level signals implying that the incumbent is overwhelmingly popular and vice-versa.

Finally, the Normal model highlights even better than the Uniform model the contrast between the rigidity of the equilibrium outcome when the incumbent’s popularity is above $\theta^*$ and the resounding defeats that occur as the incumbent’s popularity crosses below $\theta^*$. This contrast is most pronounced when $\sigma^2$ is small and is illustrated in Figure 4, which plots the equilibrium election results and levels of fraud at $\sigma^2 = \frac{1}{100}$ as a dashed black line. We denote the associated equilibrium popularity threshold by $\theta^*_2$. We see that when each agent has an almost perfect information about the incumbent’s national-level popularity $\theta$, shifts in the agents’ perception of the incumbent’s popularity result in a herd-like coordination: On the one hand, virtually all agents conduct fraud on behalf of the incumbent regardless of the actual value of $\theta$ as long as $\theta > \theta^*$; on the other hand, the minor shift in the incumbent’s popularity from just above to just below $\theta^*$ results in a defeat by the margin of $F\%$ as virtually all agents change their behavior from conducting fraud to refraining from it.

3 Empirical Analysis

[EMPIRICS HERE; PLEASE SEE THE ATTACHED SLIDES]
References


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York: Free Press.


Anomalous Results:
Last Digits for Putin
Anomalous Results:
Relationship between digits and vote share

|      | Estimate | Std. Error | t value | Pr(>|t|) |
|------|----------|------------|---------|----------|
| (Intercept) | 68.4015  | 0.1843     | 371.19  | 0.0000   |
| 0.5   | -3.5671  | 0.2712     | -13.15  | 0.0000   |
| 1     | -3.1066  | 0.2692     | -11.54  | 0.0000   |
| 1.5   | -3.6354  | 0.2702     | -13.45  | 0.0000   |
| 2     | -3.2222  | 0.2706     | -11.91  | 0.0000   |
| 2.5   | -3.2392  | 0.2731     | -11.86  | 0.0000   |
| 3     | -3.0493  | 0.2712     | -11.24  | 0.0000   |
| 3.5   | -3.4870  | 0.2725     | -12.79  | 0.0000   |
| 4     | -2.8227  | 0.2699     | -10.46  | 0.0000   |
| 4.5   | -2.3452  | 0.2703     | -8.68   | 0.0000   |
| 5     | -0.4173  | 0.2636     | -1.58   | 0.1134   |
| 5.5   | -2.0154  | 0.2695     | -7.48   | 0.0000   |
| 6     | -1.6043  | 0.2677     | -5.99   | 0.0000   |
| 6.5   | -2.3534  | 0.2714     | -8.67   | 0.0000   |
| 7     | -1.2483  | 0.2689     | -4.64   | 0.0000   |
| 7.5   | -2.7124  | 0.2717     | -9.98   | 0.0000   |
| 8     | -0.9867  | 0.2671     | -3.69   | 0.0002   |
| 8.5   | -2.3294  | 0.2718     | -8.57   | 0.0000   |
| 9     | -1.6329  | 0.2688     | -6.07   | 0.0000   |
| 9.5   | -1.8394  | 0.2696     | -6.82   | 0.0000   |

Dependent variable is the percent vote for Putin in a precinct.
Anomalous Results in the Russian Presidential Election, 2012
Anomalous Results in the Russian Presidential Election, 2012
Anomalous Results:
95% and 99% CIs Based on Other Candidates
Anomalous Results:
95% CIs of the Kernel Density Estimate

Anomalous Ruggedness = Empirical Density - KDE

95% CI (right)
95% CI (left)
Anomalous Results:
95% CIs of the Kernel Density Estimate

Absolute Value of Anomalous Ruggedness (above 95% CI)
Who Was Putin Stealing From?
Supplementary Appendix to “Deliver the Vote!
Micromotives and Macrobehavior in Electoral Fraud”

This appendix contains proofs of those technical results that do not follow directly from the discussion in the text.

A.1 The Posterior Density of $\theta|S_i$

Our assumption that $\theta$ is uniformly distributed on the unit interval $(0, 1)$ implies that the probability density of $\theta$, $f(\theta)$, is

$$f(\theta) = \begin{cases} 1, & \text{if } 0 < \theta < 1; \\ 0, & \text{otherwise}. \end{cases}$$

Similarly, our assumption that $S_i$ is uniformly distributed on the interval $(\theta - \epsilon, \theta + \epsilon)$ implies that

$$f(S_i|\theta) = \begin{cases} \frac{1}{2\epsilon}, & \text{if } \theta - \epsilon < S_i < \theta + \epsilon; \\ 0, & \text{otherwise}. \end{cases}$$

Using Bayes’ rule for random variables, we see that $g(\theta|S_i)$, the posterior density of $\theta$ given that agent $i$ observes the incumbent’s precinct-level popularity $S_i$, is

$$g(\theta|S_i) = \frac{f(S_i|\theta)f(\theta)}{f(S_i)} \text{ where } f(S_i) = \int_{-\infty}^{\infty} f(S_i|\theta)f(\theta) d\theta.$$ 

Because the support of $f(\theta)$ is limited to $(0, 1)$, while the density $f(S_i|\theta)$ implies that $S_i$ may take values on $(-\epsilon, 1+\epsilon)$, we need to account for the lower and upper bounds 0 and 1 on the integration limits in the computation of $f(S_i)$ when $S_i$ is within an $\epsilon$ distance of these bounds. That is, if $\theta$ is further than an $\epsilon$ distance from the boundaries 0 or 1,
\[ \epsilon < S_i < 1 - \epsilon, \text{ then} \]

\[ f(S_i) = \int_{S_i - \epsilon}^{S_i + \epsilon} \frac{1}{2\epsilon} \, d\theta = 1 \quad \text{and} \quad g(\theta|S_i) = \frac{1}{2\epsilon}. \]

Meanwhile, if \( \theta \) is within an \( \epsilon \) distance of 0, \( -\epsilon < S_i < \epsilon \), then

\[ f(S_i) = \int_0^{S_i + \epsilon} \frac{1}{2\epsilon} \, d\theta = 1 \quad \text{and} \quad g(\theta|S_i) = \frac{1}{S_i + \epsilon}, \]

whereas, if \( \theta \) is within an \( \epsilon \) distance of 1, \( 1 - \epsilon < S_i < 1 + \epsilon \), then

\[ f(S_i) = \int_{S_i - \epsilon}^1 \frac{1}{2\epsilon} \, d\theta = 1 \quad \text{and} \quad g(\theta|S_i) = \frac{1}{1 - (S_i + \epsilon)}. \]

Lemma 1.

\[ \theta|S_i \sim \begin{cases} 
\text{Uniform}(0, S_i + \epsilon) & \text{if } -\epsilon < S_i \leq \epsilon; \\
\text{Uniform}(S_i - \epsilon, S_i + \epsilon) & \text{if } \epsilon < S_i < 1 - \epsilon; \\
\text{Uniform}(S_i - \epsilon, 1) & \text{if } 1 - \epsilon \leq S_i < 1 + \epsilon.
\end{cases} \]

Proof. Follows from the text. \( \square \)

A.2 The Upper Bounds on \( F \) and \( \epsilon \)

Lemma 1 implies that our claim in the main text that the posterior density of \( \theta|S_i \) is uniform on the interval \( (S_i - \epsilon, S_i + \epsilon) \) holds as long as the signals \( S_i \) come from the interval \( (\epsilon, 1 - \epsilon) \). The interval on which the signals \( S_i \) are relevant for the global game analysis in the main text is \( (\frac{1}{2} - F - \epsilon, \frac{1}{2} + \epsilon) \). Hence, we must have

\[ \epsilon < \frac{1}{2} - F - \epsilon \quad \text{or equivalently} \quad \epsilon < \frac{1}{4}(1 - 2F) \quad \text{and} \quad F < \frac{1}{2}(1 - 4\epsilon). \]
In turn, the maximum admissible values of $F$ and $\epsilon$ are

$$\bar{\epsilon} = \frac{1}{4}(1 - 2F) \quad \text{and} \quad \overline{F} = \frac{1}{2}(1 - 4\epsilon),$$

with $\epsilon > 0$ and $F > 0$ implying $\bar{\epsilon} < \frac{1}{4}$ and $\overline{F} < \frac{1}{2}$.

$F < \frac{1}{2}$ is also required for the existence of the left strict-dominance region in our global game. That is, the region $0 < \theta < \frac{1}{2} - F$ in which all agents strictly prefer to refrain from fraud (c.f. Morris and Shin 2003, 65).

### A.3 Comparative Statics

The thresholds $S^*$ are $\theta^*$ are decreasing in $F$ and $w$ since

$$\frac{\partial S^*}{\partial F} = \frac{\partial \theta^*}{\partial F} = -\frac{w}{c + w} < 0, \quad \frac{\partial S^*}{\partial w} = -\frac{c(F + 2\epsilon)}{(c + w)^2} < 0 \quad \text{and} \quad \frac{\partial \theta^*}{\partial w} = -\frac{cF}{(c + w)^2} < 0;$$

and $S^*$ are $\theta^*$ are increasing in $c$

$$\frac{\partial S^*}{\partial c} = \frac{w(F + 2\epsilon)}{(c + w)^2} > 0 \quad \text{and} \quad \frac{\partial \theta^*}{\partial c} = \frac{wF}{(c + w)^2} > 0.$$

The effect of $\epsilon$ on $S^*$ depends on whether $w > c$ since

$$\frac{\partial S^*}{\partial \epsilon} = \frac{c - w}{c + w}.$$

### A.4 The Normal Model

Figure 1 illustrates the information structure of the Normal Model by plotting the common prior about the incumbent’s popularity $\theta$ (dashed line), $N(\theta_0', \sigma_0^2)$ with $\theta_0' = 0, \sigma_0^2 = 1$, against the threshold agent’s posterior belief $\theta|S_i$ about the incumbent’s national-level
Figure 1: The common prior $\mathcal{N}(0, 1)$ (dashed line) and the threshold agent’s posterior belief about the incumbent’s national-level popularity after observing the signal $S_i = S_i^*$ (solid line) for $\sigma^2 = \frac{1}{4}$ (left) and $\sigma^2 = \frac{1}{100}$ (right). The mean $\frac{\sigma^2 S_i^* + \sigma^2 \theta_0^*}{\sigma_0^2 + \sigma^2}$ of the posterior density $\theta | S_i$ is denoted by $\theta^P$.

In order to find the equilibrium fraud, popularity, and majority thresholds, it will be useful to rewrite these quantities and the equilibrium conditions from the main text in terms of the probit-transformed popularity $\theta'$ and the agents’ signals $S_i'$. The majority threshold becomes

$$\phi^* = \frac{\frac{1}{2} - \Phi(\theta')}{F};$$

(A.1)

the fraction of agents $\phi$ that engage in fraud in equilibrium corresponds to one minus the cumulative distribution function of the $\mathcal{N}(\theta', \sigma^2)$ density evaluated at $S_i'$,

$$\phi = 1 - \Phi\left(\frac{S_i' - \theta'}{\sigma}\right);$$

(A.2)

and the threshold agent’s belief that the incumbent will lose the election (from the
indifference condition) becomes

$$\Pr \left[ R < \frac{1}{2} \mid S_i = S^* \right] = \Pr \left[ \phi < \phi^* \mid S_i = S^* \right] = \Pr \left[ \phi < \frac{1}{2} - \Phi(\theta') \mid S_i = S^* \right] = \Pr \left[ \Phi(\theta') < \frac{1}{2} - \phi F \mid S_i = S^* \right] = \Pr \left[ \theta' < \Phi^{-1} \left( \frac{1}{2} - \phi F \right) \mid S_i = S^* \right],$$

which corresponds to the cumulative distribution function of the $\mathcal{N} \left( \frac{\sigma_0^2 S_i' + \sigma^2 \theta_0'}{\sigma_0^2 + \sigma^2}, \frac{\sigma_0^2 \sigma^2}{\sigma_0^2 + \sigma^2} \right)$ density evaluated at $\Phi^{-1} \left( \frac{1}{2} - \phi F \right)$,

$$\Pr \left[ R < \frac{1}{2} \mid S_i = S^* \right] = \Phi \left( \frac{\Phi^{-1} \left( \frac{1}{2} - \phi F \right) - \frac{\sigma_0^2 S_i' + \sigma^2 \theta_0'}{\sigma_0^2 + \sigma^2}}{\sqrt{\frac{\sigma_0^2 \sigma^2}{\sigma_0^2 + \sigma^2}}} \right). \tag{A.3}$$

Updating the indifference condition using (A.3) and combining with (A.2), we see that in equilibrium

$$\phi = \frac{1}{2} - \Phi(\theta^*'), \tag{A.4}$$

$$\Phi \left( \frac{\Phi^{-1} \left( \frac{1}{2} - \phi F \right) - \frac{\sigma_0^2 S_i' + \sigma^2 \theta_0'}{\sigma_0^2 + \sigma^2}}{\sqrt{\frac{\sigma_0^2 \sigma^2}{\sigma_0^2 + \sigma^2}}} \right) = \frac{w}{c + w}, \tag{A.5}$$

where $\phi = 1 - \Phi \left( \frac{S_i' - \theta^*'}{\sigma} \right)$ according to (A.2). Equilibrium condition (A.4) states that, when $S_i' = S^*$ and $\theta' = \theta^*$, the fraction of agents that receive a signal of at least $S^*$ is exactly the fraction of agents needed to deliver a bare majority to the incumbent.

Equilibrium condition (A.5) states that the threshold agent with the signal $S_i' = S^*$ is indifferent between engaging in fraud and refraining from it.
This set of two equations about two unknowns can be reduced to a single equation in \( \theta' \) by solving (A.4) for \( S^{*'} \),

\[
S^{*'} = \theta^{*'} + \sigma \Phi^{-1} \left( 1 - \frac{\frac{1}{2} - \Phi(\theta^{*'})}{F} \right),
\]

(A.6)

and substituting it into (A.5), (after some algebra) we obtain

\[
\frac{\sigma_0}{\sqrt{\sigma_0^2 + \sigma^2}} \Phi^{-1} \left( 1 - \frac{\frac{1}{2} - \Phi(\theta^{*'})}{F} \right) = \frac{\sigma}{\sigma_0 \sqrt{\sigma_0^2 + \sigma^2}} (\theta^{*'} - \theta_0) - \Phi^{-1} \left( \frac{w}{c + w} \right).
\]

(A.7)

Equilibrium condition (A.7) has a unique solution for \( \theta^{*'} \) as long as the signal \( S_i \) is sufficiently precise relative to the prior belief about \( \theta \). That is, as long as \( \sigma^2 \) is not too large relative to \( \sigma_0^2 \). Observe that the right-hand-side of (A.7) is linearly increasing in \( \theta^{*'} \) with the slope \( \frac{\sigma}{\sigma_0 \sqrt{\sigma_0^2 + \sigma^2}} \). Meanwhile, the left-hand-side of (A.7) is also increasing in \( \theta^{*'} \) but it is non-linear (it is inverse S-shaped along the y-axis) and its slope may be smaller than \( \frac{\sigma}{\sigma_0 \sqrt{\sigma_0^2 + \sigma^2}} \) for large values of \( \sigma^2 \) (relative to \( \sigma_0^2 \)).

In these cases, (A.7) may no longer have a unique solution.

Figure 2 illustrates the two cases. More formally, the partial derivative of the left-hand-side of (A.7) with respect to \( \theta' \) is

\[
\frac{\partial LHS}{\partial \theta'} = \frac{\sigma_0}{\sqrt{\sigma_0^2 + \sigma^2}} \frac{e^{-\frac{\theta'^2}{2}} + \text{erfc}^{-1} \left[ 2 - 2 \frac{\theta'}{F} \right]^2}{F} > 0,
\]

where \( \text{erfc}^{-1}(x) \) is the inverse complementary error function (see e.g. Olver et al. 2010, 160),

\[
\text{erfc}^{-1}(x) = 1 - \text{erf}(x) \quad \text{and} \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.
\]

\( ^1 \)The left-hand-side is defined only on the interval \( \Phi^{-1}(\frac{1}{2} - F) < \theta' < 0 \), with \( \lim_{\theta' \to 0} = \infty \) and \( \lim_{\theta' \to \Phi^{-1}(\frac{1}{2} - F)} = -\infty \).
Figure 2: The left-hand-side versus the right-hand-side of (A.7) for the case of a unique equilibrium (left) and a case that lacks uniqueness (right)

In turn, the slope of the left-hand-side of 2 is steeper than the slope of its right-hand-side as long as

$$\frac{\sigma}{\sigma_0} < \min_{\theta'} \frac{e^{-\frac{\theta'^2}{2}} + \text{erfc}^{-1}\left[2 - \frac{1 - 2\Phi(\theta')}{F}\right]^2}{F}.$$  

The above is a sufficient condition for a unique solution to (A.7). For the parameters values used in the paper, it implies that uniqueness is guaranteed as long as $\sigma < 4.84$.

Finally, the comparative statics of $S^*$ are $\theta^*$ with respect to $F$, $w$, and $c$ remain identical to the Uniform model. Differentiating (A.7) with respect to $w$, the left-hand-side is zero (since it does not depend on $w$) while the right-hand-side is negative as

$$\frac{\partial \text{RHS}}{\partial w} = -\sqrt{2\pi} c e^{\text{erfc}^{-1}\left[\frac{2w}{c+w}\right]^2} \frac{(c+w)^2}{(c+w)^2} < 0.$$  

This implies that an increase in $w$ shifts the right-hand-side (A.7) downward while the left-hand-side is unchanged, resulting in a decrease in $\theta^{*'}$ (since the left-hand-side is
increasing in $\theta'$. A similar argument confirms that $\theta^{*'}$ increasing in $c$, since

$$\frac{\partial \text{RHS}}{\partial c} = \frac{\sqrt{2\pi} \, w \, e^{\text{erfc}^{-1}\left[\frac{2w}{c+w}\right]^2}}{(c+w)^2} > 0.$$ 

Meanwhile, $\theta^{*'}$ is decreasing in $F$, as the partial derivative of the right-hand-side with respect to $F$ is zero (since it does not depend on $F$) while the derivative of the left-hand-side is positive as

$$\frac{\partial \text{LHS}}{\partial F} = -\frac{\sigma_0 \sqrt{\frac{\pi}{2}} \text{erf} \left[ \frac{\theta'}{\sqrt{2}} \right] e^{\text{erfc}^{-1}\left[2 \frac{\text{erf} \left[ \frac{\theta'}{\sqrt{2}} \right]}{F} \right]^2}}{F^2 \sqrt{\sigma_0^2 + \sigma^2}} > 0,$$

where $\text{erf} \left[ \frac{\theta'}{\sqrt{2}} \right] < 0$. The same relationship with respect to $F$, $w$, and $c$ holds for $S^*$ since $S^{*'}$ is increasing in $\theta^{*'}$ according to (A.6).

References
